

## ABSOLUTELY SUMMING MULTILINEAR OPERATORS VIA INTERPOLATION

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**ABSTRACT.** We use an interpolative technique from [1] to introduce the notion of multiple  $N$ -separately summing operators. Our approach extends and unifies some recent results; for instance we recover the best known estimates of the multilinear Bohnenblust-Hille constants due to F. Bayart, D. Pellegrino and J. Seoane-Sepúlveda. More precisely, as a consequence of our main result, for  $1 \leq t < 2$  and  $m > 1$  we prove that

$$\left( \sum_{i_1, \dots, i_m=1}^{\infty} |U(e_{i_1}, \dots, e_{i_m})|^{\frac{2tm}{2+(m-1)t}} \right)^{\frac{2+(m-1)t}{2tm}} \leq \left[ \prod_{j=2}^m \Gamma \left( 2 - \frac{2-t}{jt-2t+2} \right)^{\frac{t(j-2)+2}{2t-2jt}} \right] \|U\|$$

for all complex  $m$ -linear forms  $U : c_0 \times \dots \times c_0 \rightarrow \mathbb{C}$ .

## 1. INTRODUCTION AND PRELIMINARIES

We use standard notations and notions from Banach space theory as, *e.g.*, in [5]. The Banach spaces  $X_1, \dots, X_m, X, Y$  are considered over the scalar field  $\mathbb{K}$ , with  $\mathbb{K}$  be  $\mathbb{R}$  or  $\mathbb{C}$ . A continuous linear operator between Banach spaces  $u : X \rightarrow Y$  is absolutely summing when  $(\|u(x_j)\|)_{j \in \mathbb{N}} \in \ell_1$  whenever  $(x_j)_{j \in \mathbb{N}}$  is unconditionally summable. The theory of absolutely summing operators has its origins in the 50's with Grothendieck's resumé but only in 1966-67 that the class of summing operators was presented in its modern form (see [5, 10, 14] for more details).

The success of the linear theory of absolutely summing operators motivated the emergence of a non linear theory. In 1983 A. Pietsch [15] initiated a research program sketching the roots of the multilinear theory. Now, the multilinear theory of absolutely summing operators is a very fruitful field of nonlinear Functional Analysis with important connections with other fields. We stress, for instance, the striking advances in the estimates of the Bohnenblust-Hille constants and its applications to the final solution of the optimal estimate of the Bohr radius [2, 3] and in quantum information theory [11].

Let  $2 \leq q < \infty$ . A Banach space  $X$  has *cotype*  $q$  if there is a constant  $C > 0$  such that, no matter how we select finitely many vectors  $x_1, \dots, x_n \in X$ ,

$$(1.1) \quad \left( \sum_{k=1}^n \|x_k\|^q \right)^{\frac{1}{q}} \leq C \left( \int_I \left\| \sum_{k=1}^n r_k(t)x_k \right\|^2 dt \right)^{\frac{1}{2}},$$

where  $I := [0, 1]$  and  $r_k$  denotes the  $k$ -th Rademacher function. The smallest of all these constants is denoted by  $C_q(X)$  and it is called the cotype  $q$  constant of  $X$ . In fact, up to the constant  $C$  the definition of cotype can be changed by replacing the  $L_2$  norm by an  $L_p$  norm in (1.1). More precisely:

**Theorem 1.1** (Kahane Inequality). *Let  $0 < p, q < \infty$ . Then there is a constant  $K_{p,q} > 0$  for which*

$$\left( \int_I \left\| \sum_{k=1}^n r_k(t)x_k \right\|^q dt \right)^{\frac{1}{q}} \leq K_{p,q} \left( \int_I \left\| \sum_{k=1}^n r_k(t)x_k \right\|^p dt \right)^{\frac{1}{p}},$$

*holds, regardless of the choice of a Banach space  $X$  and of finitely many vectors  $x_1, \dots, x_n \in X$ .*

The previous theorem is a generalization of the Khinchine inequality, which holds for  $q = 2$  and  $X = \mathbb{K}$ . In this case the optimal constants are known and denoted by  $A_p^{\mathbb{K}}$ . For real scalars, U. Haagerup [7] proved that

$$(1.2) \quad A_p^{\mathbb{R}} = \frac{1}{\sqrt{2}} \left( \frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}} \right)^{-\frac{1}{p}}, \quad \text{for } 1.85 \approx p_0 < p < 2$$

and

$$(1.3) \quad A_p^{\mathbb{R}} = 2^{\frac{1}{p}-\frac{1}{2}}, \quad \text{for } 1 \leq p \leq p_0 \approx 1.85.$$

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The exact definition of  $p_0$  is the following:  $p_0 \in (1, 2)$  is the unique real number satisfying

$$\Gamma\left(\frac{p_0 + 1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

For complex scalars, H. König [8] (see also [9]) using Steinhaus variables instead of Rademacher functions has shown that

$$(1.4) \quad A_p^C = \left(\Gamma\left(\frac{p+2}{2}\right)\right)^{-\frac{1}{p}} \quad \text{for } 1 \leq p < 2.$$

The weak  $\ell_1$ -norm of vectors  $x_1, \dots, x_n$  in a Banach space  $X$  is defined by

$$\|(x_i)_{i=1}^n\|_{w,1} := \sup_{\|x'\|_{X'} \leq 1} \sum_{i=1}^n |x'(x_i)|.$$

From now on  $X, X_1, \dots, X_m, Y$  will denote Banach spaces. By  $\mathcal{L}(X_1, \dots, X_m; Y)$  denote the Banach space of all (bounded)  $m$ -linear operators  $U : X_1 \times \dots \times X_m \rightarrow Y$ . For  $1 \leq r < \infty$ ,  $U \in \mathcal{L}(X_1, \dots, X_m; Y)$  is called *multiple  $(r, 1)$ -summing*, if there exists a constant  $C > 0$  such that

$$\left( \sum_{i_1, \dots, i_m=1}^n \|U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)})\|_Y^r \right)^{\frac{1}{r}} \leq C \prod_{k=1}^m \left\| (x_i^{(k)})_{i=1}^n \right\|_{w,1}$$

for all finite choice of vectors  $x_i^{(k)} \in X_k$ ,  $1 \leq i \leq n$ ,  $1 \leq k \leq m$ . The vector space of all multiple  $(r, 1)$ -summing operators is denoted by  $\Pi_{(r,1)}^m(X_1, \dots, X_m; Y)$ . The infimum,  $\pi_{(r,1)}^m(U)$ , taken over all possible constants  $C$  satisfying the previous inequality defines a complete norm in  $\Pi_{(r,1)}^m(X_1, \dots, X_m; Y)$ .

We need to recall some useful multi-index notation: for a positive integer  $n$  and a finite subset  $D \subset \mathbb{N}$ , we denote by  $|D|$  the cardinality of  $D$  and define the index set

$$\mathcal{M}(D, n) := \left\{ \mathbf{i} = (i_k)_{k \in D} \in \mathbb{N}^{|D|}; i_k \in \{1, \dots, n\} \text{ for each } k \in D \right\}.$$

Futher,  $\mathcal{P}_k(D)$  will denote the set of subsets of  $D$  with cardinality  $k$ , for  $1 \leq k \leq |D|$ . When  $D = \{1, \dots, m\}$ , we will simply write

$$\mathcal{M}(m, n) := \mathcal{M}(\{1, \dots, m\}, n) = \{\mathbf{i} = (i_1, \dots, i_m) \in \mathbb{N}^m; i_1, \dots, i_m \in \{1, \dots, n\}\}$$

and

$$\mathcal{P}_k(m) := \mathcal{P}_k(\{1, \dots, m\}).$$

For  $S = \{s_1, \dots, s_k\} \in \mathcal{P}_k(m)$ , its complement will be  $\widehat{S} := \{1, \dots, m\} \setminus S$  and  $\mathbf{i}_S$  shall mean  $(i_{s_1}, \dots, i_{s_k}) \in \mathcal{M}(k, n)$ .

The following well-known lemmata will be useful along this paper (we refer to [4, Lemma 2.2] and [6, Corollary 5.4.2]):

**Lemma 1.2.** *Let  $X$  be a cotype  $q$  Banach space,  $1 \leq r \leq q$  and  $(x_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}(m,n)}$  be a matrix in  $X$ . Then*

$$\left( \sum_{\mathbf{i} \in \mathcal{M}(m,n)} \|x_{\mathbf{i}}\|_X^q \right)^{\frac{1}{q}} \leq C_q(X)^m K_{r,2}^m \left( \int_{I^m} \left\| \sum_{\mathbf{i} \in \mathcal{M}(m,n)} r_{\mathbf{i}}(t) x_{\mathbf{i}} \right\|^r dt \right)^{\frac{1}{r}}$$

where, for each  $\mathbf{i} = (i_1, \dots, i_m) \in \mathcal{M}(m, n)$ ,  $r_{\mathbf{i}}(t) = r_{i_1}(t_1) \cdots r_{i_m}(t_m)$  and  $dt = dt_1 \dots dt_m$ .

**Lemma 1.3.** *For  $0 < p < q < +\infty$ , and any sequence of scalars  $(a_{ij})_{i,j \in \mathbb{N}}$  we have*

$$\left( \sum_i \left( \sum_j |a_{ij}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq \left( \sum_j \left( \sum_i |a_{ij}|^q \right)^{\frac{p}{q}} \right)^{\frac{1}{p}}.$$

## 2. THE INTERPOLATIVE APPROACH

We now recall the interpolative approach introduced in [1] that was crucial (see [2]) to obtain the ultimate constants of the Bohehnblust-Hille inequalities and to provide the precise asymptotic growth of the Bohr radius.

For  $\mathbf{p} = (p_1, \dots, p_m) \in [1, +\infty)^m$ , and a Banach space  $X$ , we shall consider the space

$$\ell_{\mathbf{p}}(X) := \ell_{p_1}(\ell_{p_2}(\dots(\ell_{p_m}(X))\dots)),$$

namely, a vector matrix  $(x_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}(m, \mathbb{N})} \in \ell_{\mathbf{p}}(X)$  if, and only if,

$$\left( \sum_{i_1=1}^{\infty} \left( \sum_{i_2=1}^{\infty} \left( \dots \left( \sum_{i_{m-1}=1}^{\infty} \left( \sum_{i_m=1}^{\infty} \|x_{\mathbf{i}}\|_X^{p_m} \right)^{\frac{p_{m-1}}{p_m}} \right)^{\frac{p_{m-2}}{p_{m-1}}} \dots \right)^{\frac{p_2}{p_3}} \right)^{\frac{p_1}{p_2}} \right)^{\frac{1}{p_1}} < +\infty.$$

When  $X = \mathbb{K}$ , we just write  $\ell_{\mathbf{p}}$  instead of  $\ell_{\mathbf{p}}(\mathbb{K})$ . The core of the interpolative approach from [1] is summarized as follows (we sketch the proof for the sake of completeness):

**Lemma 2.1** (Interpolation procedure). *Let  $m, n, N$  be positive integers and  $\mathbf{q}, \mathbf{q}(1), \dots, \mathbf{q}(N) \in [1, +\infty)^m$  be such that  $(\frac{1}{q_1}, \dots, \frac{1}{q_m})$  belongs to the convex hull of  $(\frac{1}{q_1(k)}, \dots, \frac{1}{q_m(k)})$ ,  $k = 1, \dots, N$ . Then for all scalar matrix  $\mathbf{a} = (a_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}(m, n)}$ ,*

$$\|\mathbf{a}\|_{\mathbf{q}} \leq \prod_{k=1}^N \|\mathbf{a}\|_{\mathbf{q}(k)}^{\theta_k},$$

i.e.,

$$\left( \sum_{i_1=1}^n \left( \dots \left( \sum_{i_m=1}^n |a_{\mathbf{i}}|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \dots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq \prod_{k=1}^N \left[ \left( \sum_{i_1=1}^n \left( \dots \left( \sum_{i_m=1}^n |a_{\mathbf{i}}|^{q_m(k)} \right)^{\frac{q_{m-1}(k)}{q_m(k)}} \dots \right)^{\frac{q_1(k)}{q_2(k)}} \right)^{\frac{1}{q_1(k)}} \right]^{\theta_k},$$

where  $\theta_k$  are the coordinates of  $(\frac{1}{q_1}, \dots, \frac{1}{q_m})$  on the convex hull.

*Sketch of the proof.* We just follows the lines of [1, Proposition 2.1]. Proceeding by induction on  $N$  and using that, for any Banach space  $X$  and  $\theta \in [0, 1]$ ,

$$\ell_{\mathbf{r}}(X) = [\ell_{\mathbf{p}}(X), \ell_{\mathbf{q}}(X)]_{\theta},$$

with  $\frac{1}{r_i} = \frac{\theta}{p_i} + \frac{1-\theta}{q_i}$ , for  $i = 1, \dots, m$ . If

$$\frac{1}{q_i} = \frac{\theta_1}{q_i(1)} + \dots + \frac{\theta_N}{q_i(N)},$$

with  $\sum_{k=1}^N \theta_k = 1$  and each  $\theta_k \in [0, 1]$ , then we also have

$$\frac{1}{q_i} = \frac{\theta_1}{q_i(1)} + \frac{1-\theta_1}{p_i},$$

setting

$$\frac{1}{p_i} = \frac{\alpha_2}{q_i(2)} + \dots + \frac{\alpha_N}{q_i(N)}, \quad \text{and } \alpha_j = \frac{\theta_j}{1-\theta_1},$$

for  $i = 1, \dots, m$  and  $j = 2, \dots, N$ . So  $\alpha_j \in [0, 1]$  and  $\sum_{j=2}^N \alpha_j = 1$ . Therefore, by the induction hypothesis, we conclude that

$$\|\mathbf{a}\|_{\mathbf{q}} \leq \|\mathbf{a}\|_{\mathbf{q}(1)}^{\theta_1} \cdot \|\mathbf{a}\|_{\mathbf{p}}^{1-\theta_1} \leq \|\mathbf{a}\|_{\mathbf{q}(1)}^{\theta_1} \cdot \left[ \prod_{j=2}^N \|\mathbf{a}\|_{\mathbf{q}(j)}^{\alpha_j} \right]^{1-\theta_1} = \prod_{k=1}^N \|\mathbf{a}\|_{\mathbf{q}(k)}^{\theta_k}.$$

□

Consequently, combining the previous result with Lemma 1.3 the following generalization of the Blei inequality arises (see [2, Remark 2.2]):

**Lemma 2.2** (Bayart, Pellegrino, Seoane-Sepulveda). *Let  $m, n$  be positive integers,  $1 \leq k \leq m$  and  $1 \leq s \leq q$ . Then for all scalar matrix  $(a_i)_{i \in \mathcal{M}(m,n)}$ ,*

$$\left( \sum_{i \in \mathcal{M}(m,n)} |a_i|^\rho \right)^{\frac{1}{\rho}} \leq \prod_{S \in \mathcal{P}_k(m)} \left( \sum_{i_S} \left( \sum_{i_{\bar{S}}} |a_i|^q \right)^{\frac{s}{q}} \right)^{\frac{1}{s} \cdot \frac{1}{\binom{m}{k}}},$$

where

$$\rho := \frac{msq}{kq + (m-k)s}.$$

### 3. MULTIPLE SUMMING OPERATORS WITH MULTIPLE EXPONENTS

In this section we apply the interpolation procedure to generalize results of the theory of multiple summing multilinear operators. Our main result recovers, with a new approach, one of the main results of [4].

For Banach spaces  $X_1, \dots, X_m$  and a proper non-void subset  $D \subset \{1, \dots, m\}$  let  $X^D$  be the product  $\prod_{k \in D} X_k$ . A vector  $x_D \in X^D$  may be seen as an element  $\widetilde{x}_D = (\widetilde{x}_D^1, \dots, \widetilde{x}_D^m) \in X_1 \times \dots \times X_m$ , with  $\widetilde{x}_D^i = x_D^i$ , if  $i \in D$ , and  $\widetilde{x}_D^i = 0$ , otherwise. Given  $U \in \mathcal{L}(X_1, \dots, X_m; Y)$ , we define the map

$$\begin{aligned} U^D : X^{\widehat{D}} &\rightarrow \mathcal{L}(X^D; Y) \\ x_{\widehat{D}} &\mapsto U_{x_{\widehat{D}}}^D : X^D \rightarrow Y \\ &\quad y_D \mapsto U(\widetilde{x}_{\widehat{D}} + \widetilde{y}_D). \end{aligned}$$

Clearly  $U^D$  is well-defined and  $|\widehat{D}|$ -linear. Notice that, for each  $x_{\widehat{D}} \in X^{\widehat{D}}$ ,  $U_{x_{\widehat{D}}}^D$  is the restriction of  $U$  to the  $D$ -coordinates, with the  $\widehat{D}$ -coordinates fixed through  $x_{\widehat{D}}$ . The following definition was introduced in [4].

**Definition 3.1.** *Let  $1 \leq r < \infty$  and let  $D$  be a proper subset of  $\{1, \dots, m\}$ . We say that  $U \in \mathcal{L}(X_1, \dots, X_m; Y)$  is multiple  $(r, 1)$ -summing in the coordinates of  $D$  (or multiple  $(r, 1)$ -summing in  $D$ ) whenever  $U^D$  has its range in  $\Pi_{(r,1)}^{|\widehat{D}|}(X^D; Y)$ . Moreover,  $U$  is separately  $(r, 1)$ -summing if  $U$  is multiple  $(r, 1)$ -summing in all one point subset of  $\{1, \dots, m\}$ .*

The following result came from a careful look at the argument in the proof of [4, Theorem 4.1]. It provides estimates for bounded  $m$ -linear operators that are multiple  $(r, 1)$ -summing in the coordinates of a fixed index proper subset of  $\{1, \dots, m\}$ .

**Theorem 3.2** (Defant, Popa, Schwarting). *Let  $Y$  be a cotype  $q$  Banach space,  $1 \leq r \leq q$  and suppose that  $D \subseteq \{1, \dots, m\}$  is non-void and proper. If  $U \in \mathcal{L}(X_1, \dots, X_m; Y)$  is multiple  $(r, 1)$ -summing in the coordinates of  $D$ , then*

$$\left( \sum_{i_D} \left( \sum_{i_{\widehat{D}}} \|U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)})\|^q \right)^{\frac{r}{q}} \right)^{\frac{1}{r}} \leq A_{q,r}^{|\widehat{D}|}(Y) \|U^D : X^{\widehat{D}} \rightarrow \Pi_{(r,1)}^{|\widehat{D}|}(X^D; Y)\|$$

for all finite choice of vectors  $x_1^{(k)}, \dots, x_N^{(k)} \in X_k$ , with  $\left\| (x_j^{(k)})_{j=1}^N \right\|_{w,1} \leq 1$ , for  $k = 1, \dots, m$  and  $A_{q,r}(Y) := C_q(Y)K_{r,2}$ .

Above and from now on, the symbol  $\sum_{i_D}$  means that we are taking the sum over the indices  $i_k$ , with  $k \in D$ . Also the constant  $A_{q,r}(Y)$  is defined as above. The main result of this section reads as follows:

**Theorem 3.3.** *Let  $Y$  be a cotype  $q$  Banach space,  $1 \leq r_1, \dots, r_n \leq q$  and  $\{1, \dots, m\}$  be the disjoint union of non-void proper subsets  $C_1, \dots, C_n$ . If  $U \in \mathcal{L}(X_1, \dots, X_m; Y)$  is multiple  $(r_k, 1)$ -summing in each coordinate subset  $C_k$ , for  $k = 1, \dots, n$ , then*

$$\begin{aligned} &\left( \sum_{i_{C_1}} \left( \sum_{i_{C_2}} \left( \dots \left( \sum_{i_{C_n}} \|U(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)})\|_Y^{q_n} \right)^{\frac{q_{n-1}}{q_n}} \dots \right)^{\frac{q_2}{q_3}} \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \\ &\leq \prod_{k=1}^n \left[ A_{q,r_k}^{|\widehat{C_k}|}(Y) \|U^{C_k} : X^{\widehat{C_k}} \rightarrow \Pi_{r_k,1}^{|\widehat{C_k}|}(X^{C_k}; Y)\| \right]^{\theta_k}, \end{aligned}$$

regardless of the finite choice of vectors  $x_1^{(l)}, \dots, x_N^{(l)} \in X_l$  with  $\left\| \left( x_j^{(l)} \right)_{j=1}^N \right\|_{w,1} \leq 1$ ,  $l = 1, \dots, m$ . Here, each  $q_k \in [r_k, q]$  is such that  $\frac{1}{q_k} = \frac{\theta_k}{r_k} + \frac{(1-\theta_k)}{q}$ , with  $\theta_1, \dots, \theta_n \in [0, 1]$  and  $\sum_{k=1}^n \theta_k = 1$ .

*Proof.* Since  $U$  is multiple  $(r_k, 1)$ -summing in each subset  $C_k$ , the previous theorem assures that, for  $x_1^{(l)}, \dots, x_N^{(l)} \in X_l$  with  $\left\| \left( x_j^{(l)} \right)_{j=1}^N \right\|_{w,1} \leq 1$ ,  $l = 1, \dots, m$ ,

$$\left( \sum_{\mathbf{i}_{C_k}} \left( \sum_{\mathbf{i}_{\widehat{C_k}}} \left\| U \left( x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)} \right) \right\|_Y^q \right)^{\frac{r_k}{q}} \right)^{\frac{1}{r_k}} \leq A_{q,r_k}^{|\widehat{C_k}|}(Y) \left\| U^{C_k} : X^{\widehat{C_k}} \rightarrow \Pi_{(r_k,1)}^{|\widehat{C_k}|}(X^{C_k}; Y) \right\|.$$

for  $k = 1, \dots, n$ . Now, Lemma 1.3 guarantees that we may change the position of the exponents  $r_k$  and  $q$  (with the correspondent indices):

$$\begin{aligned} & \left( \sum_{\mathbf{i}_{C_1}, \dots, \mathbf{i}_{C_{k-1}}} \left( \sum_{\mathbf{i}_{C_k}} \left( \sum_{\mathbf{i}_{C_{k+1}}, \dots, \mathbf{i}_{C_n}} \left\| U \left( x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)} \right) \right\|_Y^q \right)^{\frac{r_k}{q}} \right)^{\frac{q}{r_k}} \right)^{\frac{1}{q}} \\ & \leq \left( \sum_{\mathbf{i}_{C_k}} \left( \sum_{\mathbf{i}_{\widehat{C_k}}} \left\| U \left( x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)} \right) \right\|_Y^q \right)^{\frac{r_k}{q}} \right)^{\frac{1}{r_k}} \\ & \leq A_{q,r_k}^{|\widehat{C_k}|}(Y) \left\| U^{C_k} : X^{\widehat{C_k}} \rightarrow \Pi_{(r_k,1)}^{|\widehat{C_k}|}(X^{C_k}; Y) \right\|. \end{aligned}$$

On the other hand, the hypotheses on  $q_1, \dots, q_m$  mean precisely that  $\left( \frac{1}{q_1}, \dots, \frac{1}{q_m} \right)$  belongs to the convex hull of the points  $\left( \frac{1}{q_1(k)}, \dots, \frac{1}{q_m(k)} \right)$ ,  $k = 1, \dots, n$ , with

$$q_j(k) := \begin{cases} r_k, & \text{if } j \in C_k; \\ q, & \text{if } j \in \widehat{C_k}, \end{cases}$$

for  $k = 1, \dots, n$ . Therefore, the interpolation method of Lemma 2.1 gives us

$$\begin{aligned} & \left( \sum_{\mathbf{i}_{C_1}} \left( \sum_{\mathbf{i}_{C_2}} \left( \dots \left( \sum_{\mathbf{i}_{C_n}} \left\| U \left( x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)} \right) \right\|_Y^{q_n} \right)^{\frac{q_2}{q_3}} \dots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \right)^{\theta_k} \\ & \leq \prod_{k=1}^n \left[ \left( \sum_{\mathbf{i}_{C_1}, \dots, \mathbf{i}_{C_{k-1}}} \left( \sum_{\mathbf{i}_{C_k}} \left( \sum_{\mathbf{i}_{C_{k+1}}, \dots, \mathbf{i}_{C_n}} \left\| U \left( x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)} \right) \right\|_Y^q \right)^{\frac{r_k}{q}} \right)^{\frac{q}{r_k}} \right)^{\frac{1}{q}} \right]^{\theta_k} \\ & \leq \prod_{k=1}^n \left[ A_{q,r_k}^{|\widehat{C_k}|}(Y) \left\| U^{C_k} : X^{\widehat{C_k}} \rightarrow \Pi_{(r_k,1)}^{|\widehat{C_k}|}(X^{C_k}; Y) \right\| \right]^{\theta_k}. \end{aligned}$$

□

As a particular case of this result, we obtain one of the main results of [4]. Before, we need to recall some technical definitions (see [4, Section 3]): for  $q \geq 2$ , let us consider the functions  $w, f : [1, q]^2 \rightarrow [0, +\infty)$  defined by

$$\omega(x, y) := \frac{q^2(x+y) - 2qxy}{q^2 - xy} \quad \text{and} \quad f(x, y) := \frac{q^2x - qxy}{q^2(x+y) - 2qxy}.$$

Inductively, one may define  $w_n : [1, q]^n \rightarrow [0, +\infty)$  by  $w_2(x_1, x_2) := \omega(x_1, x_2)$ , and, for  $n \geq 3$ ,

$$w_n(x_1, \dots, x_n) := w_2(x_n, w_{n-1}(x_1, \dots, x_{n-1})).$$

We proceed similarly for  $f_n := (f_n^1, \dots, f_n^n) : [1, q]^n \rightarrow [0, +\infty)^n$ . First,  $f_2(x_1, x_2) := (f(x_1, x_2), f(x_2, x_1))$ . Inductively, the function  $f_n$  (in  $n$  variables  $x_1, \dots, x_n$ ) is defined using  $f_{n-1}$  (in the  $n-1$  variables  $x_1, \dots, x_{n-1}$ ) by

$$f_n^k(x_1, \dots, x_n) := f_{n-1}^k(x_1, \dots, x_{n-1}) \cdot f(\omega_{n-1}(x_1, \dots, x_{n-1}), x_n), \quad k = 1, \dots, n-1,$$

and

$$f_n^n(x_1, \dots, x_n) := f(x_n, \omega_{n-1}(x_1, \dots, x_{n-1})).$$

For any choice of  $(x_1, \dots, x_n) \in [1, q]^n$ , it can be checked by induction that

$$\sum_{k=1}^n f_n^k(x_1, \dots, x_n) = 1.$$

Now let us see how to recover the main result of [4] from theorem 3.3.

**Corollary 3.4.** *Let  $\{1, \dots, m\}$  be the disjoint union of non-void proper subsets  $C_1, \dots, C_n$ , let  $Y$  be a Banach space with cotype  $q$ , and suppose that  $1 \leq r_1, \dots, r_n < q$ . Assume that  $U \in \mathcal{L}(X_1, \dots, X_m; Y)$  is multiple  $(r_k, 1)$ -summing in each  $C_k$ ,  $1 \leq k \leq n$ . Then  $U$  is multiple  $(\omega_n, 1)$ -summing, and*

$$\pi_{(\omega_n, 1)}^m(U) \leq \sigma_n \prod_{k=1}^n \left\| U^{C_k} : X^{\widehat{C_k}} \rightarrow \Pi_{r_k, 1}^{|C_k|}(X^{C_k}; Y) \right\|^{f_n^k},$$

where  $\sigma_n$  is defined by

$$\sigma_2 = \left( A_{q, r_1}^{|C_2|}(Y) \right)^{f(r_1, r_2)} \left( A_{q, r_2}^{|C_1|}(Y) \right)^{f(r_2, r_1)},$$

and for  $n \geq 3$

$$\sigma_n = \left( A_{q, r_n}^{|\cup_{k=1}^{n-1} C_k|}(Y) \right)^{f(r_n, \omega_{n-1})} \left( A_{q, \omega_{n-1}}^{|C_n|}(Y) \right)^{f(\omega_{n-1}, r_n)} \sigma_{n-1}^{f(\omega_{n-1}, r_n)}.$$

*Proof.* Using the following formulas for the exponents  $\omega_n := \omega_n(r_1, \dots, r_n)$  and  $f_n^k := f_n^k(r_1, \dots, r_n)$  (see [16, Theorem 3.2])

$$\omega_n = \frac{qR}{1+R} \quad \text{and} \quad f_n^k = \frac{r_k}{R(q-r_k)}, \quad k = 1, \dots, n, \quad \text{where} \quad R := \sum_{k=1}^n \frac{r_k}{q-r_k},$$

and taking  $\theta_k := f_n^k$ ,  $k = 1, \dots, n$ , in theorem 3.3, we get

$$\frac{1}{q_k} = \frac{1}{R(q-r_k)} + \frac{1}{q} \left( 1 - \frac{r_k}{R(q-r_k)} \right) = \frac{1+R}{qR} = \frac{1}{\omega_n},$$

for  $k = 1, \dots, n$ . Thus theorem 3.3 guarantees that  $U$  is  $(\omega_n, 1)$ -summing and

$$\pi_{(\omega_n, 1)}^m(U) \leq \prod_{k=1}^n \left[ A_{q, r_k}^{\widehat{C_k}}(Y) \right]^{f_n^k} \cdot \left\| U^{C_k} : X^{\widehat{C_k}} \rightarrow \Pi_{r_k, 1}^{|C_k|}(X^{C_k}; Y) \right\|^{f_n^k}.$$

This is precisely the result stated, up to the constants  $\sigma_n$  for  $n \geq 3$ . In order to recover these, one need to proceed by induction as described in the proof of [4, Theorem 4.2], using that  $U$  is multiple  $(\omega_{n-1}, 1)$ -summing in the coordinates of  $\cup_{k=1}^{n-1} C_k$ , and by assumption that  $U$  also it is multiple  $(r_n, 1)$ -summing in the coordinates of  $C_n$ .  $\square$

The following important special case is highlighted in [4, Section 3] as an immediate consequence of the previous result.

**Corollary 3.5** ([4, Section 3]; Corollary 5.2). *Let  $Y$  be a Banach space with cotype  $q$ , and  $1 \leq r < q$ . Then there is a constant  $\sigma_m \geq 1$  such that each separately  $(r, 1)$ -summing operator  $U \in \mathcal{L}(X_1, \dots, X_m; Y)$  is multiple  $\left( \frac{qrm}{q+(m-1)r}, 1 \right)$ -summing, and*

$$\pi_{\left( \frac{qrm}{q+(m-1)r}, 1 \right)}^m(U) \leq \sigma_m \prod_{k=1}^m \left\| U^{\{k\}} : X^{\widehat{\{k\}}} \rightarrow \Pi_{r, 1}(X^{\{k\}}; Y) \right\|^{\frac{1}{m}}$$

where  $\sigma_m$ , as stated in Corollary 3.4, depends on  $m$ ,  $r$ ,  $q$  and  $C_q(Y)$ .

In the next section, we show that the previous result is a particular case of an even more general theorem.

4. MULTIPLE  $N$ -SEPARATE SUMMABILITY

The following definition is a variation of Definition 3.1.

**Definition 4.1.** Let  $1 \leq r < \infty$ . We say that  $U \in \mathcal{L}(X_1, \dots, X_m; Y)$  is  $N$ -separately  $(r, 1)$ -summing, when  $U$  is multiple  $(r, 1)$ -summing in each subset of  $\{1, \dots, m\}$  with cardinality  $N$ .

In other words,  $U \in \mathcal{L}(X_1, \dots, X_m; Y)$  is  $N$ -separately  $(r, 1)$ -summing if  $U$  is multiple  $(r, 1)$ -summing in  $S \subseteq \{1, \dots, m\}$ , for all  $S \in \mathcal{P}_N(m)$ . In this context,  $U$  is separately  $(r, 1)$ -summing if and only if  $U$  is 1-separately  $(r, 1)$ -summing.

From now on  $Y$  is a Banach space with cotype  $q$ . The following result extends Corollary 3.5:

**Theorem 4.2.** Let  $1 \leq r \leq q$ , and  $1 \leq n < m$ . If  $U \in \mathcal{L}(X_1, \dots, X_m; Y)$  is  $n$ -separately  $(r, 1)$ -summing, then  $U$  is  $N$ -separately  $(r_N, 1)$ -summing, for all  $n < N \leq m$ , with  $r_N := \frac{qrN}{nq + (N-n)r}$ . Moreover, if  $N < m$ , we get, for each  $D \in \mathcal{P}_N(m)$ ,

$$\pi_{(r_N, 1)}^N(U_{x_{\widehat{D}}}^D) \leq A_{q, r}^{N-n}(Y) \prod_{S \in \mathcal{P}_n(D)} \left\| (U_{x_{\widehat{D}}}^D)^S : X^{D \setminus S} \rightarrow \Pi_{(r, 1)}^n(X^S; Y) \right\|^{(\frac{1}{N})}$$

for all  $x_{\widehat{D}} \in X^{\widehat{D}}$ . The estimate for  $N = m$  becomes

$$\pi_{(r_m, 1)}^m(U) \leq A_{q, r}^{m-n}(Y) \prod_{S \in \mathcal{P}_n(m)} \left\| U^S : X^{\widehat{S}} \rightarrow \Pi_{(r, 1)}^n(X^S; Y) \right\|^{(\frac{1}{m})}.$$

*Proof.* Firstly, we will prove the result for  $n < N < m$ . Let  $D \in \mathcal{P}_N(m)$ . Without loss of generality, we may assume that  $D = \{1, \dots, N\}$ . We must prove that  $U^D$  has its range in  $\Pi_{(r_N, 1)}^{|D|}(X^D; Y)$ , that is, given  $x_{\widehat{D}} \in X^{\widehat{D}}$ ,  $U_{x_{\widehat{D}}}^D \in \Pi_{(r_N, 1)}^{|D|}(X^D; Y)$ . Clearly,  $U_{x_{\widehat{D}}}^D$  is bounded and  $N$ -linear. On the other hand, since  $U$  is  $n$ -separately  $(r, 1)$ -summing,  $U_{x_{\widehat{D}}}^D$  is  $n$ -separately  $(r, 1)$ -summing, i.e.,  $U_{x_{\widehat{D}}}^D$  is  $(r, 1)$ -summing in  $S \subset D$ , for all  $S \in \mathcal{P}_n(D) = \mathcal{P}_n(N)$ . Let  $M$  be a positive integer and  $x_1^{(k)}, \dots, x_M^{(k)} \in X_k$  be such that  $\left\| (x_j^{(k)})_{j=1}^M \right\|_{w, 1} \leq 1$ , for  $k = 1, \dots, N$ . Also, let us set  $x_{\mathbf{i}} := (x_{i_1}^{(1)}, \dots, x_{i_N}^{(N)}) \in X^D$ , for  $\mathbf{i} = (i_1, \dots, i_N) \in \mathcal{M}(N, M) = \{1, \dots, M\}^N$ . Lemma 2.2 implies

$$\left( \sum_{\mathbf{i}} \|U_{x_{\widehat{D}}}^D(x_{\mathbf{i}})\|^{r_N} \right)^{\frac{1}{r_N}} \leq \prod_{S \in \mathcal{P}_n(N)} \left( \sum_{\mathbf{i}_S} \left( \sum_{\mathbf{i}_{D \setminus S}} \|U_{x_{\widehat{D}}}^D(x_{\mathbf{i}})\|^q \right)^{\frac{r}{q}} \right)^{\frac{1}{r} \cdot \frac{1}{(N)}} ,$$

with the sum  $\sum_{\mathbf{i}}$  taken over all multi-index  $\mathbf{i} = (i_1, \dots, i_N) \in \mathcal{M}(N, M)$ . Since  $U_{x_{\widehat{D}}}^D$  is  $n$ -separately  $(r, 1)$ -summing, Theorem 3.2 assures that

$$\left( \sum_{\mathbf{i}_S} \left( \sum_{\mathbf{i}_{D \setminus S}} \|U_{x_{\widehat{D}}}^D(x_{\mathbf{i}})\|^q \right)^{\frac{r}{q}} \right)^{\frac{1}{r}} \leq A_{q, r}^{N-n}(Y) \left\| (U_{x_{\widehat{D}}}^D)^S : X^{D \setminus S} \rightarrow \Pi_{(r, 1)}^n(X^S; Y) \right\|.$$

Therefore,

$$\left( \sum_{\mathbf{i}} \|U_{x_{\widehat{D}}}^D(x_{\mathbf{i}})\|^{r_N} \right)^{\frac{1}{r_N}} \leq A_{q, r}^{N-n}(Y) \prod_{S \in \mathcal{P}_n(N)} \left\| (U_{x_{\widehat{D}}}^D)^S : X^{D \setminus S} \rightarrow \Pi_{(r, 1)}^n(X^S; Y) \right\|^{(\frac{1}{N})},$$

and this conclude the result for  $n < N < m$ . For  $N = m$ , one just need to work with the maps  $U^S : X^{\widehat{S}} \rightarrow \Pi_{(r, 1)}^n(X^S; Y)$ , for each  $S \in \mathcal{P}_n(m)$ , and follows the lines of the previous argument.  $\square$

Notice that if  $U$  is 1-separately  $(r, 1)$ -summing, then it is  $N$ -separately  $(\frac{qrN}{q + (N-1)r}, 1)$ -summing for all  $1 \leq N \leq m$ . To recover Corollary 3.5, i.e.,  $U \in \mathcal{L}(X_1, \dots, X_m; Y)$  is multiple  $(\frac{qrm}{q + (m-1)r}, 1)$ -summing,  $U$  just need to be  $n$ -separately  $(s, 1)$ -summing for some  $1 \leq n < m$  and  $s \leq \frac{qrn}{q + (n-1)r}$ .

We observe that, in some special cases, our approach provides better exponents. In fact, let  $1 < n < N \leq m$  and suppose that  $U$  is  $n$ -separately  $(r, 1)$ -summing. Let  $k, l \in \mathbb{N}$ , with  $l < n$ , be such that  $N = kn + l$ . Thus, if

$l \neq 0$ , given  $S \in \mathcal{P}_N(m)$ , we may choose  $C_1, \dots, C_k \in \mathcal{P}_n(m)$  and  $C_{k+1} \in \mathcal{P}_l(m)$  such that

$$\bigcup_{j=1}^{k+1} C_j = S$$

with this union be disjoint. Clearly, since  $l < n$  we conclude that  $U$  is multiple  $(r, 1)$ -summing in the coordinates of  $C_{k+1}$  and (using the hypothesis)  $U$  is multiple  $(r, 1)$ -summing in the coordinates of  $C_j$  for  $1 \leq j \leq k$ . Since  $\omega_{k+1}(r, \dots, r) = \frac{q(k+1)r}{q+kr}$ , using [4, Theorem 5.1] and the arbitrariness of  $S \in \mathcal{P}_N(m)$ , one may conclude that  $U$  is  $N$ -separately  $\left(\frac{q(k+1)r}{q+kr}, 1\right)$ -summing. Nevertheless, Theorem 4.2 assures that  $U$  is  $N$ -separately  $\left(\frac{qrN}{nq+(N-n)r}, 1\right)$ -summing. Note that, since  $l \neq 0$ ,

$$\frac{q(k+1)r}{q+kr} > \frac{qrN}{nq+(N-n)r}.$$

If  $l = 0$ , we will obtain that  $\omega_k(r, \dots, r) = \frac{qrk}{q+(k-1)r} = \frac{qrN}{nq+(N-n)r}$ . Therefore, the exponent provided by Theorem 4.2 is more efficient.

As a final remark we note that Theorem 4.2 is also useful to provide estimates for the constants involved. For instance, if we take  $X_1 = \dots = X_m = c_0$  and  $Y = \mathbb{K}$ , we obtain better estimates to the constants of some variation of Bohnenblust-Hille inequalities introduced in [12, Appendix A] and [13]. More precisely, it shows that for all parameters  $1 \leq t < 2$  and all  $m \in \mathbb{N}$ , there exists a constant  $C_{m,t}^{\mathbb{K}} \geq 1$ , such that,

$$\left( \sum_{i_1, \dots, i_m=1}^{\infty} |U(e_{i_1}, \dots, e_{i_m})|^{\frac{2tm}{2+(m-1)t}} \right)^{\frac{2+(m-1)t}{2tm}} \leq C_{m,t}^{\mathbb{K}} \|U\|,$$

for all  $m$ -linear forms  $U : c_0 \times \dots \times c_0 \rightarrow \mathbb{K}$ , with

$$(4.1) \quad C_{m,t}^{\mathbb{K}} = \begin{cases} 1 & \text{if } m = 1, \\ \left( A_{\frac{2mt}{(m-2)t+4}}^{\mathbb{K}} \right)^{\frac{m}{2}} C_{\frac{m}{2},t}^{\mathbb{K}} & \text{if } m \text{ is even, and} \\ \left( \left( A_{\frac{2(m-1)t}{(m-3)t+4}}^{\mathbb{K}} \right)^{\frac{m+1}{2}} C_{\frac{m-1}{2},t}^{\mathbb{K}} \right)^{\frac{m-1}{2m}} \left( \left( A_{\frac{2(m+1)t}{(m-1)t+4}}^{\mathbb{K}} \right)^{\frac{m-1}{2}} C_{\frac{m+1}{2},t}^{\mathbb{K}} \right)^{\frac{m+1}{2m}} & \text{if } m \text{ is odd.} \end{cases}$$

This can be easily inserted in the context of multiple multilinear forms: for each parameter  $t \in [1, 2)$ , we have a coincidence result for  $m$ -linear maps

$$\mathcal{L}(c_0, \dots, c_0; \mathbb{K}) = \Pi_{\left(\frac{2tm}{2+(m-1)t}, 1\right)}^m(c_0, \dots, c_0; \mathbb{K}),$$

which means that every bounded  $m$ -linear forms  $U : c_0 \times \dots \times c_0 \rightarrow \mathbb{K}$  is multiple  $\left(\frac{2tm}{2+(m-1)t}, 1\right)$ -summing. Moreover, the following norm estimates holds:

$$\pi_{\left(\frac{2tm}{2+(m-1)t}, 1\right)}^m(U) \leq C_{m,t}^{\mathbb{K}} \|U\|.$$

Combining this with Theorem 4.2, the following estimates for the variants of Bohnenblust-Hille inequality arises.

**Theorem 4.3.** *Let  $1 \leq t < 2$  and  $m > 1$ . Then*

$$\left( \sum_{i_1, \dots, i_m=1}^{\infty} |U(e_{i_1}, \dots, e_{i_m})|^{\frac{2tm}{2+(m-1)t}} \right)^{\frac{2+(m-1)t}{2tm}} \leq C_{m,t}^{\mathbb{K}} \|U\|,$$

for all bounded  $m$ -linear forms  $U : c_0 \times \dots \times c_0 \rightarrow \mathbb{K}$ , with

$$C_{m,t}^{\mathbb{C}} \leq \prod_{j=2}^m \Gamma \left( 2 - \frac{2-t}{2+t(j-2)} \right)^{-\frac{2+t(j-2)}{2t(j-1)}},$$

and

$$C_{m,t}^{\mathbb{R}} \leq \begin{cases} 2^{\left(\frac{1}{t}-\frac{1}{2}\right) \cdot \sum_{j=1}^{m-1} \frac{1}{j}}, & \text{if } m \leq \frac{2p_0+2t(1-p_0)}{t(2-p_0)}; \\ \left[ \prod_{j=2}^{m_0} 2^{\frac{t+2m_0-2tm_0+m+jm_0-jm-2}{2t(m_0-1)(j-1)}} \right] \cdot \left[ \prod_{j=m_0+1}^m \left( \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{3}{2} - \frac{2-t}{2+t(j-2)} \right) \right)^{\frac{t(j-2)+2}{2t-2jt}} \right], & \text{if } m > \frac{2p_0+2t(1-p_0)}{t(2-p_0)}; \end{cases}$$

where  $m_0$  is the largest integer not greater than  $\frac{2p_0+2t(1-p_0)}{t(2-p_0)}$ .



*Proof.* In our context we have that, for  $t \in [1, 2)$  and  $m \geq 1$ , every bounded  $m$ -linear forms  $U : c_0 \times \cdots \times c_0 \rightarrow \mathbb{K}$  is  $n$ -separately  $\left(\frac{2tn}{2+(n-1)t}, 1\right)$ -summing, for all  $1 \leq n \leq m$ . Thus, by considering  $n = m - 1$  and using that  $U$  is  $(m-1)$ -separately  $\left(\frac{2t(m-1)}{2+(m-2)t}, 1\right)$ -summing, we invoke Theorem 4.2 to conclude that  $U$  is multiple  $\left(\frac{2tm}{2+(m-1)t}, 1\right)$ -summing and

$$\pi_{\left(\frac{2tm}{2+(m-1)t}, 1\right)}^m(U) \leq A_{2, \frac{2t(m-1)}{2+(m-2)t}}(\mathbb{K}) \prod_{S \in P_{m-1}(m)} \left\| U^S : X^{\hat{S}} \rightarrow \Pi_{\left(\frac{2t(m-1)}{2+(m-2)t}, 1\right)}^{m-1}(X^S; \mathbb{K}) \right\|^{\frac{1}{\binom{m}{m-1}}}.$$

Since for  $Y = \mathbb{K}$ , we can use  $A_{\frac{2t(m-1)}{2+(m-2)t}}^{\mathbb{K}}$  instead of  $A_{2, \frac{2t(m-1)}{2+(m-2)t}}(\mathbb{K})$ , and

$$\begin{aligned} \left\| U^S : X^{\hat{S}} \rightarrow \Pi_{\left(\frac{2t(m-1)}{2+(m-2)t}, 1\right)}^{m-1}(X^S; \mathbb{K}) \right\| &= \sup_{x \in B_{X^{\hat{S}}}} \pi_{\left(\frac{2t(m-1)}{2+(m-2)t}, 1\right)}^{m-1}(U_x^S) \\ &\leq C_{m-1, t}^{\mathbb{K}} \sup_{x \in B_{X^{\hat{S}}}} \|U_x^S\| \\ &\leq C_{m-1, t}^{\mathbb{K}} \|U\|, \end{aligned}$$

we get

$$\pi_{\left(\frac{2tm}{2+(m-1)t}, 1\right)}^m(U) \leq A_{\frac{2t(m-1)}{2+(m-2)t}}^{\mathbb{K}} C_{m-1, t}^{\mathbb{K}} \|U\|.$$

Thus,

$$C_{m, t}^{\mathbb{K}} \leq A_{\frac{2t(m-1)}{2+(m-2)t}}^{\mathbb{K}} C_{m-1, t}^{\mathbb{K}}.$$

Proceeding by induction and using that  $C_{1, t}^{\mathbb{K}} = 1$ , we obtain

$$C_{m, t}^{\mathbb{K}} = \begin{cases} 1, & \text{if } m = 1; \\ \prod_{k=1}^{m-1} A_{\frac{2tk}{2+(k-1)t}}^{\mathbb{K}}, & \text{if } m > 1. \end{cases}$$

Finally, using (1.2), (1.3) and (1.4), we obtain the result.  $\square$

By considering  $t = 1$ , we recover the Bohnenblust-Hille constants presented in [2, Proposition 3.1] and a direct calculation shows that the above theorem improves (4.1). Proceeding as in [2, Corollary 3.2 and Corollary 3.3], we have an alternative formula that highlights the asymptotic behavior of the constants.

**Theorem 4.4.** *For any  $t \in [1, 2)$ , there exists  $\kappa_{t, \mathbb{K}} > 0$  such that, for any  $m \geq 1$ ,*

$$C_{m, t}^{\mathbb{C}} \leq \kappa_{t, \mathbb{C}} m^{\frac{(\gamma-1)(t-2)}{2t}},$$

and

$$C_{m, t}^{\mathbb{R}} \leq \kappa_{t, \mathbb{R}} m^{\frac{(\gamma-2+\ln 2)(t-2)}{2t}}.$$

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